Irregularities of distribution. I

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1. Introduction

The concept of *uniformly distributed sequences* plays a fundamental role in many branches of mathematics (ergodic theory, diophantine approximation, numerical integration, mathematical statistics, etc.). The object of the theory of Irregularities of Distribution is to measure the uniformity (or nonuniformity) of sequences and point distributions. For instance: how uniformly can an arbitrary distribution of n points in the unit cube be distributed relative to a given family of "nice" sets (e.g., boxes with sides parallel to the coordinate axes, balls, convex sets, etc.)?

This theory was initiated by the following conjecture of van der Corput. Let $\zeta=\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\ldots\}$ be an infinite sequence of real numbers in the unit interval $U=[0,1]$. Given an x in U and a positive integer n, write $Z_n[\zeta;x]$ for the number of integers j with $1\leqslant j<n$ and $0\leqslant \mathbf{z}_j<x$ and put

$$D_n[\zeta;x]=Z_n[\zeta;x]-n\cdot x.$$

Let $\Delta_n[\zeta]$ be the supremum of $|D_n[\zeta;x]|$ over all numbers x in U. In 1935 van der Corput [6] conjectured that $\Delta_n[\zeta]$ cannot remain bounded as n tends to infinity. It was proved by Mrs T. van Aardenne-Ehrenfest [1] in 1945. Later her beautiful theorem was improved and extended in various directions by the work of K. F. Roth and Wolfgang M. Schmidt. There is now a vast literature on this subject. We refer the reader to Schmidt's book [13].

In this paper we continue the research started in Schmidt [11], [12]. We recall one of his basic results (Corollary of Theorem A3 in [12]): Let there be given n points