Borel selectors for upper semi-continuous set-valued maps

by

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§ 1. Introduction

A set-valued map F from a topological space X to a topological space Y is said to be *upper semi-continuous,* if the set $\{x: F(x) \cap H \neq \emptyset\}$ is closed in X, whenever H is a closed set in Y. A point-valued function f is said to be a *selector* for such a set-valued map F, if $f(x) \in F(x)$, for all x in X. The function f from X to Y is said to be a Borel measurable function of the *first Borel class* if $f^{-1}(H)$ is a $\mathscr{G}_\delta$-set in X, whenever H is a closed set in Y. Similarly, f is said to be a Borel measureable function of the *second Borel class* if $f^{-1}(H)$ is an $\mathscr{F}_{\sigma\delta}$-set in X, whenever H is a closed set in Y. In [18, Theorems 2 and 3] we prove that, if X and Y are metric spaces and F is an upper semi-continuous set-valued map from X to Y, taking only non-empty values, then F always has a Borel measureable selector of the second Borel class, further, if F only takes non-empty complete values in Y, then F always has a Borel measurable selector of the first Borel class.

Some of the more interesting upper semi-continuous set-valued maps are defined on a subset of a Banach space X and take their values in a Banach space Y with its weak topology, or in a dual Banach space Y^* with its weak-star topology. In [19, Theorem 2] we prove that if F is a weak upper semi-continuous set-valued map, defined on a metric space X, and taking only non-empty weakly compact values, contained in a fixed weakly σ-compact set of a Banach space Y, then F has a weak Borel measurable selector of the first Borel class, which is also a norm Borel measurable selector of the second Borel class. Similarly, see the introduction to [19], if F is a weak-star upper semi-continuous set-valued map, defined on a metric space X, and taking only non-empty, weak-star closed values in the dual space Y^* of a weakly compactly generated