Acta Math., 171 (1993), 231–261

# $\mathbf{HP}^2$-bundles and elliptic homology

by

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## 1. Introduction

The (universal) elliptic genus [L1] is a ring homomorphism

$$\phi: \Omega_*^{SO} \to M_* = \mathbf{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon]$$

from the oriented bordism ring to the graded polynomial ring $M_*$. Here $\delta = \phi(\mathbf{CP}^2)$ and $\varepsilon = \phi(\mathbf{HP}^2)$, where $\mathbf{CP}^2$ (resp. $\mathbf{HP}^2$) is the complex (resp. quaternionic) projective plane (an introduction and background information on elliptic genera can be found in [HBJ], [L1], [O2], [Se], [W]). The elliptic genus provides a connection between bordism theory, modular forms and quantum field theory. For, $M_*$ can be identified with a ring of modular forms and, following Witten [W], the elliptic genus $\phi(M)$ of a spin manifold $M$ can be interpreted as the $S^1$-equivariant index of an operator on the loop space on $M$. In fact, Witten used this interpretation to provide a heuristic proof for the rigidity of the elliptic genus. A rigorous proof along those lines was given by Taubes [T] (see also [BT]). The rigidity is equivalent to the multiplicativity of $\phi$ for certain fibre bundles $E \to B$ [O3]; namely, if $E$, $B$ are closed oriented manifolds, the fibre $F$ is a spin manifold and the structure group of the bundle is compact and connected then $\phi(E) = \phi(F)\phi(B)$.

The universal elliptic genus makes $M_*$ and hence $M_*[\omega^{-1}]$ for any $\omega \in M_*$ a left module over $\Omega_*^{SO}$ (recall that $M_*[\omega^{-1}] = \varinjlim M_*$, where the connecting maps in the sequence are given by multiplication by $\omega$). Landweber, Ravenel and Stong [LRS], [L1] showed that the functor

$$X \mapsto \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}} M_*[\omega^{-1}] \tag{1.1}$$

is a homology theory if $\omega = \varepsilon$ or $\omega = \delta^2 - \varepsilon$. Recently Franke [Fr] proved this for a general $\omega$ of positive degree. This 8-periodic homology theory is called (odd primary) periodic elliptic homology.

(1) The second author was partially supported by NSF Grant DMS 88-02481, the Max-Planck-Institut in Bonn and the SFB in Göttingen.