SUBALGEBRAS OF L^∞ CONTAINING H^∞

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1. Introduction

Let H^∞ be the algebra of bounded analytic functions on $D=\{z: |z|<1\}$ and let L^∞ be the Banach algebra of bounded measurable functions on $T=\{z: |z|=1\}$ with the uniform norm. Then H^∞ can be regarded as a uniformly closed subalgebra of L^∞ by identifying each $f\in H^\infty$ with its boundary function.

If A is a closed subalgebra of L^∞, let $\mathfrak{M}[A]$ denote its maximal-ideal space. K. Hoffman [13] has shown that each $\varphi\in\mathfrak{M}[H^\infty]$ has a unique norm-preserving extension to a bounded linear functional on L^∞. For example, if $z\in D$ then evaluation at z is an element of $\mathfrak{M}[H^\infty]$ and its extension is given simply by the Poisson kernel. Now if A is a closed subalgebra of L^∞ containing H^∞, then the usual Gelfand topology on $\mathfrak{M}[A]$ agrees with the weak-* topology that $\mathfrak{M}[A]$ inherits as a compact subset of the dual space of L^∞. Consequently, each $f\in L^\infty$ is continuous on $\mathfrak{M}[H^\infty]$ and harmonic on D. Moreover, if A and B are closed algebras such that $H^\infty\subset A\subset B\subset L^\infty$, then $\mathfrak{M}[H^\infty]\supset\mathfrak{M}[A]\supset\mathfrak{M}[B]\supset\mathfrak{M}[L^\infty]$. Our main result is the following theorem:

THEOREM 1. *Let A be a closed subalgebra of L^∞ containing H^∞. Let A_I be the closed subalgebra of A generated by H^∞ and $\{f^{-1}\in A: f\in H^\infty\}$. Then $\mathfrak{M}[A_I]=\mathfrak{M}[A]$.*

When combined with a recent result of S. Y. Chang [7], Theorem 1 proves a conjecture of R. Douglas [9]. To state Douglas' conjecture, we let Q be a subset of L^∞ and write $[H^\infty, Q]$ for the uniformly closed subalgebra of L^∞ generated by H^∞ and Q. An algebra of the form $[H^\infty, Q]$, where $Q\subset\{u: |u|=1 \text{ a.e. on } T \text{ and } \bar{u}\in H^\infty\}$ is called a *Douglas algebra*. Since each positive function in $(L^\infty)^{-1}$ is the modulus of a function in $(H^\infty)^{-1}$, we see that A_I is a Douglas algebra whenever $H^\infty\subset A\subset L^\infty$ and we see that if A is a Douglas algebra, then $A=A_I$. Douglas' conjecture was that every uniformly closed subalgebra A of L^∞ containing H^∞ is a Douglas algebra, or, equivalently, that every such algebra A satisfies $A=A_I$. Now S. Y. Chang has proved that if A is a closed algebra lying between H^∞ and