BOUNDARY BEHAVIOR OF A CONFORMAL MAPPING

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1. Suppose given in the complex w-plane a simply connected domain $\mathfrak{D}$, which is not the whole plane, and let $w=f(z)$ be a function mapping the open unit disc D in the z-plane one-to-one and conformally onto $\mathfrak{D}$. As is well known, for almost every θ $(0\leqslant\theta<2\pi)$, $f(z)$ has a finite *angular limit* $f(e^{i\theta})$ at $e^{i\theta}$, that is, for any open triangle Δ contained in D and having one vertex at $e^{i\theta}$, $f(z)\to f(e^{i\theta})$ as $z\to e^{i\theta}$, $z\in\Delta$. An *arc at* $e^{i\theta}$ is a curve $A\subset D$ such that $A\cup\{e^{i\theta}\}$ is a Jordan arc. As a preliminary form of our main result (Theorem 2), we state

THEOREM 1. *For almost every θ either*

$$\frac{f(z)-f(e^{i\theta})}{z-e^{i\theta}} \text{ and } f'(z) \text{ have the same finite, nonzero angular limit at } e^{i\theta}, \tag{1.1}$$

or $\arg\,(f(z)-f(e^{i\theta}))$, *defined and continuous in D, is unbounded above and below on each arc at $e^{i\theta}$.* (1.2)

Note that if (1.1) holds, the mapping is *isogonal* at $e^{i\theta}$ in the sense that

$$\arg\,(f(z)-f(e^{i\theta}))-\arg\,(z-e^{i\theta}),$$

where both argument functions are defined and continuous in D, has a finite angular limit at $e^{i\theta}$.

If $f(z)$ has a finite angular limit at $e^{i\theta}$, then the image under $f(z)$ of the radius at $e^{i\theta}$ determines an (ideal) accessible boundary point $\mathfrak{a}_\theta$ of $\mathfrak{D}$ whose complex coordinate $w(\mathfrak{a}_\theta)=f(e^{i\theta})$ is finite. The set of all such $\mathfrak{a}_\theta$ is denoted by $\mathfrak{A}$. On $\mathfrak{D}\cup\mathfrak{A}$ we use the *relative metric*, the relative distance between two points of $\mathfrak{D}\cup\mathfrak{A}$ being defined as the infimum of the Euclidean diameters of the open Jordan arcs that lie in $\mathfrak{D}$ and join these two points. Any limits involving accessible boundary points are taken in this relative metric.

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