ON THE PRINCIPLE OF SUBORDINATION IN THE THEORY OF ANALYTIC DIFFERENTIAL EQUATIONS

BY

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I. Preliminaries

1. According to Cauchy, the initial value problem

$$dw/dz = f(z, w), \quad w(0) = 0 \tag{1}$$

has a unique (regular) solution $w = w(z)$ in a neighborhood of $z = 0$ whenever f is a function of two complex variables, z and w, which is regular in a neighborhood of $(z, w) = (0,0)$. What is more, if $a > 0$ and $b > 0$ are chosen so small that $f(z, w)$ is regular on the dicylinder

$$|z| < a, \qquad |w| < b \tag{2}$$

(that is, if a convergent expansion

$$f(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} z^m w^n \tag{3}$$

holds on (2)), and if, without loss of generality, $f(z, w)$ is supposed to be bounded on (2), say

$$|f(z, w)| < M \quad \text{on} \quad (2) \qquad (M < \infty), \tag{4}$$

then there exists a $p > 0$ which depends only on the three values a, b, M and which has the property that the solution $w(z)$ of (1) exists (as a regular function) on the circle $|z| < p$. In fact, if $\|f\|$ denotes the radius of convergence of the expansion, say

$$w(z) = \sum_{n=1}^{\infty} a_n z^n, \tag{5}$$

of the solution $w(z)$ of (1), then it is known (cf. [8], pp. 127–128) that

$$\|f\| \geq \min(a, b/M). \tag{6}$$

There is an extensive literature (cf. [4], pp. 169–172), initiated by a paper of Painlevé ([6]; not quoted in [4]), which aims at an improvement of (6). I noticed however (cf. [8],