SIMULTANEOUS APPROXIMATION TO ALGEBRAIC NUMBERS BY RATIONALS

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1. Introduction

We shall prove theorems on simultaneous approximation which generalize Roth's well-known theorem [3] on rational approximation to a single algebraic irrational α.

Throughout the paper, $\|\xi\|$ will denote the distance from a real number ξ to the nearest integer.

THEOREM 1. *Let $\alpha_1, \ldots, \alpha_n$ be real algebraic numbers such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over the field Q of rationals. Then for every $\varepsilon > 0$ there are only finitely many positive integers q with*

$$\|q\alpha_1\| \cdot \|q\alpha_2\| \ldots \|q\alpha_n\| \cdot q^{1+\varepsilon} < 1. \tag{1}$$

COROLLARY. *Suppose $\alpha_1, \ldots, \alpha_n, \varepsilon$ are as above. There are only finitely many n-tuples $(p_1/q, \ldots, p_n/q)$ of rationals satisfying*

$$|\alpha_i - (p_i/q)| < q^{-1-(1/n)-\varepsilon} \qquad (i = 1, 2, \ldots, n). \tag{2}$$

A dual to Theorem 1 is as follows.

THEOREM 2. *Let $\alpha_1, \ldots, \alpha_n, \varepsilon$ be as in Theorem 1. There are only finitely many n-tuples of nonzero integers $q_1, \ldots, q_n$ with*

$$\|q_1\alpha_1 + \ldots + q_n\alpha_n\| \cdot |q_1 q_2 \ldots q_n|^{1+\varepsilon} < 1. \tag{3}$$

COROLLARY. *Again let $\alpha_1, \ldots, \alpha_n, \varepsilon$ be as in Theorem 1. There are only finitely many $(n+1)$-tuples of integers $q_1, q_2, \ldots, q_n, p$ with $q = \max(|q_1|, \ldots, |q_n|) > 0$ and with*

$$|q_1\alpha_1 + \ldots + q_n\alpha_n + p| > q^{-n-\varepsilon}. \tag{4}$$