THE RATIONAL HOMOTOPY THEORY OF CERTAIN PATH SPACES WITH APPLICATIONS TO GEODESICS

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It is well known that the topology of various path spaces on a complete riemannian manifold M is closely related to the existence of various kinds of geodesics on M. Classical Morse theory and the theory of closed geodesics are beautiful examples of this sort.

The motivation for the present paper is the study of geodesics satisfying a very general boundary condition of which the above examples and the example of isometry-invariant geodesics are particular cases. In particular, we generalize a result of Sullivan–Vigué [16].

Let $N \subset M \times M$ be a submanifold of the riemannian product $M \times M$. An *N-geodesic* on M is a geodesic $c: [0, 1] \to M$ which satisfies the boundary condition

$$(N) \qquad (c(0), c(1)) \in N \quad \text{and} \quad (\dot{c}(0), -\dot{c}(1)) \in TN^{\perp},$$

where $TN^{\perp}$ is the normal bundle of N in $M \times M$. If $N = V_1 \times V_2$, where $V_i \subset M$, $i = 1, 2$ are submanifolds of M then an N-geodesic is simply a $V_1 - V_2$ *connecting geodesic* (orthogonal to each V_i). If N is the graph of an isometry, A, of M then an N-geodesic is a geodesic which extends uniquely to an *A-invariant geodesic* $c: \mathbf{R} \to M$; i.e.

$$c(t+1) = A(c(t)), \quad t \in \mathbf{R}.$$

When A has finite order ($A^k = \mathrm{id}$) then c is in fact closed ($c(t+k) = c(t)$, $t \in \mathbf{R}$).

The study of N-geodesics on M proceeds via critical point theory for the energy integral on a suitable Hilbert manifold of curves with endpoints in N. This Hilbert manifold is homotopy equivalent to the space M_N^I of continuous curves $f: [0, 1] \to M$ satisfying $(f(0), f(1)) \in N$, with the compact open topology (cf. Grove [4], [6]).

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