Homotopy classes in Sobolev spaces and the existence of energy minimizing maps

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Introduction

Consider the problem of finding maps $f: M \to N$ that are stationary for an energy functional such as $\int_M |Df|^p$ (where M and N are compact riemannian manifolds and $p \geq 1$). Such maps may be found by minimizing the functional, but if we minimize among all maps from M to N, then the minimum is 0 and is attained only by constant maps. Thus in order to find nontrivial stationary maps, we would like to use the topology of M and N to define classes of maps from M to N in which we can minimize the functional. For instance one could try to minimize among maps in a given homotopy class, but this is not possible in general (unless $p > \dim M$), since a minimizing sequence of mappings in one homotopy class can converge (in the appropriate weak topology) to a map in another homotopy class. However, in this paper we show that it is possible to minimize among maps f whose restrictions to a lower dimensional skeleton of (a triangulation of) M belong to a given homotopy class.

To state the results precisely we need to refer to certain Sobolev spaces and norms. We will assume without loss of generality that M and N are submanifolds of euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and we let $\mathrm{Lip}(M, N)$ denote the space of lipschitz maps from M to N. We define $L^{1,p}(M, \mathbf{R}^n)$ to be the space of all functions $f \in L^p(M, \mathbf{R}^n)$ such that there exist functions

$$f_i \in \mathrm{Lip}(M, \mathbf{R}^n) \quad \text{and} \quad g \in L^p(M, \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n))$$

satisfying

$$\|f_i - f\|_p + \|Df_i - g\|_p \to 0.$$

1–888285 *Acta Mathematica* 160. Imprimé le 25 février 1988