On the invariant subspace problem for Banach spaces

by

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0. Introduction

In this paper we will present a new approach to the invariant subspace problem for Banach spaces. Our main result will be that there exists a Banach space B and an operator T on B such that T has only trivial invariant subspaces. We feel though that the ideas of the approach can be used also to prove results about existence of invariant subspaces. As an example of this, see [1]. In Section 1 we give the general ideas of the approach. In this section we also reduce the problem of proving our main result to the problem of proving Theorem 1.3. In Section 2 we prove an inequality which will be the basic tool in the construction. In Section 3 we first reduce the problem of proving Theorem 1.3 to the problem of proving 6 statements. These statements contain a parameter k. We first give lemmas and propositions which give these statements for $k=1$ and $k=2$. We then give the induction hypothesis and the lemmas and propositions which give the statements for all positive integers k. In Section 4, finally, we give proofs of Theorem 1.2 of Section 1 and of the lemmas and propositions of Section 3. An outline of this construction was presented in Enflo [2]. This version is the same—except for some changes in the presentation—as was given in Enflo [3]. The author wishes to thank professor Enrico Bombieri for suggesting these changes.

1. Outline of the proof

We will below construct an operator with only trivial invariant subspaces on a Banach space. The Banach space in this example will be constructed at the same time as the operator and will be non-reflexive. There are very serious difficulties in carrying out a similar construction in a reflexive Banach space. So we feel that the construction gives some weak support to the conjecture that every operator on a Hilbert space has a non-trivial invariant subspace. We now turn to the basic considerations behind this approach. It is clear that every operator with a cyclic vector on a Banach space can be