APPROXIMATION OF THE DIRICHLET PROBLEM ON A HALF SPACE

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There have been two general methods used most frequently to study the convergence of finite difference approximations of elliptic boundary value problems—most results in this area are based on an application of either the maximum principle or a variational principle. In this paper we attempt to develop a third approach to this problem. Our philosophy is to imitate as closely as possible the methods that have been developed to handle the differential equation itself. Of course the first step in this program is to study boundary value problems on a half space. Here we consider approximations of the Dirichlet problem on a half space H for an elliptic differential operator of arbitrary (even) order; we do *not* assume that ∂H is aligned with respect to the grid of the difference equation. For a certain class of difference schemes we give a necessary and sufficient condition for the convergence of the approximation. This condition, which involves only the symbols of the operators in the equation and not the operators themselves, is completely analogous to the so-called "covering condition" imposed on the boundary conditions of elliptic differential equations. (See for example [4], p. 125]. The accuracy of the difference schemes considered here is too limited for them to be important computationally, but we hope that our methods may serve as a first step towards a general theory for difference equations, not requiring an intermediate variational formulation and without the limitations associated with the maximum principle.

§ 1. Formulation of the results

Suppose $P(D)$ is an elliptic differential operator on $\mathbf{R}^n$, homogeneous of order $2m$, with constant real coefficients. Consider an approximation to $P(D)$ by a difference operator

$$Q_h(D) = h^{-2m} \sum_{j \in \mathbf{Z}^n} c_j \exp(ih\langle j, D\rangle) = h^{-2m} \sum_{j \in \mathbf{Z}^n} c_j T_{hj} \quad \text{(a finite sum)}$$

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