THE HOMOLOGY GROUPS OF SOME ORDERED SYSTEMS

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The considerations of this paper were suggested by the 'abelian' version of a problem concerning a Fuchsian group G. Greenberg [4] has shown that if $\mathcal{J}(G)$ denotes the system of all finitely generated subgroups of G, ordered by the relation of being included of finite index, then $\mathcal{J}(G)$ has maximal elements. To comprehend the ordered system $\mathcal{J}(G)$, a first approximation is to look at its 'homology groups', these being definable for any partially ordered set with zero (see below). The resulting problem is still intractable, and it seemed of interest to try the analogous problem when G is replaced by a finitely generated abelian group M: the analogue of the maximal elements of $\mathcal{J}(G)$ is then the family $\mathcal{S}(M)$ of all direct factors of M. Here, $\mathcal{S}(M)$ happens to be a lattice, ordered by inclusion, and we form from it a complex ΨM whose vertices are the elements of $\mathcal{S}(M)$, and whose simplices $(v_0, \ldots, v_q)$ are ordered sets of vertices such that $v_0 \cap \ldots \cap v_q \neq 0$. The homology groups of ΨM are then the ones we consider (with related matters) in this paper. A principal result (see section 15) states:

If M has $n \geqslant 3$ generators, then ΨM contains a wedge of $(n-2)$-spheres, and the inclusion induces isomorphisms of homology and homotopy groups. The set of spheres is bijective with the group of all $n \times n$ non-singular rational upper triangular matrices, modulo the diagonal matrices.

Just as ΨM was formed from the partially ordered set $\mathcal{S}(M)$, we can form a complex ΨP^n from the lattice $\mathrm{Flat}(P^n)$ of flats of a projective n-space $P^n(F)$ over a field F. It happens that $\Psi M \approx \Psi P^{n-1}(\mathbf{Q})$. When F is finite, the Möbius function μ of $\mathrm{Flat}(P^n)$ was studied by Rota [11] who related it to the Euler Characteristic χ of ΨP^n, and calculated μ and χ. Our treatment, however, is geometrical rather than arithmetical, and thus gives more information (for example, here ΨP^n has the homotopy type of a wedge of spheres.) We take a more general point of view than Rota, working at first with partially ordered sets rather than lattices, and abstracting the role of the 'support' $v_0 \cap \ldots \cap v_q$ of a simplex $(v_0, \ldots, v_q)$. It turns out (see section 16) that Rota's equation $\chi = 1 + \mu$ for a general finite