ON SOME NON-LINEAR ELLIPTIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

BY

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The object of this paper is to obtain existence and uniqueness theorems for (weak) uniformly Lipschitz continuous solutions $u(x)$ of Dirichlet boundary value problems associated with non-linear elliptic differential-functional equations of the form

$$[a_j(\operatorname{grad} u)]_{x_j} + F[u](x) = 0, \tag{0.1}$$

where, for a fixed x, $F[u](x)$ is a non-linear functional of u. The results to be obtained can be considered as generalizations of some theorems of Gilbarg [5] and Stampacchia [14] in the case $F[u] \equiv 0$ and of some theorems of Stampacchia [14] in certain cases $F[u] \not\equiv 0$.

Part I deals with the functional analysis basis for the proofs. It gives existence theorems for the solutions of certain non-linear, functional inequalities. By a weak solution of (0.1) on a domain Ω is usually understood a function $u(x)$ having a gradient u_x in some sense and satisfying

$$\int_\Omega \left\{ a_j(u_x)\eta_{x_j} - F[u]\eta \right\} dx = 0 \tag{0.2}$$

for all continuously differentiable $\eta(x)$ with compact support in Ω, i.e., $\eta \in C_0^1(\Omega)$. Part I will imply existence and uniqueness theorems for functions $u(x)$, to be called quasi solutions, satisfying

$$\int_\Omega \left\{ a_j(u_x)\eta_{x_j} - F[u]\eta \right\} dx \geqslant 0 \tag{0.3}$$

for η in certain subsets of $C_0^1(\Omega)$ depending on u. A particular case of this situation arises, for example, if one seeks the solution of a variational problem

$$\min \int_\Omega \{ f(u_x) + \dots \} dx$$

(1) This research was partially supported by the Air Force Office of Scientific Research under Contract AF 49 (638)–1382 and Grant AF EOAR 65–42.