EXTREMAL AND CONJUGATE EXTREMAL DISTANCE ON OPEN RIEMANN SURFACES WITH APPLICATIONS TO CIRCULAR-RADIAL SLIT MAPPINGS

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Partition the boundary of a compact bordered Riemann surface $\overline{W}$ into four disjoint sets $\alpha_0, \alpha, \beta, \gamma$ with α_0 and α non-empty. Let $\hat{W}$ denote the compactification of W obtained by adding to W a point for each boundary component. Define

$$F = \{c : c \text{ is an arc in } \hat{W} - \gamma \text{ from } \alpha_0 \text{ to } \alpha\}$$

and $\quad F^* = \{c : c \text{ is a sum of closed curves in } \hat{W} - \beta \text{ such that } c \text{ separates } \alpha_0 \text{ from } \alpha\}.$

Determine the harmonic function u in W by the boundary conditions $u=0$ on α_0, $u=1$ on α, $\partial u/\partial n = 0$ along γ and u is constant on each component β_i in β such that $\int_{\beta_i} du^* = 0$. Then $\lambda(F) = \|du\|^{-2}$, $\lambda(F^*) = \|du\|^2$ (see Lemma III.1.1) where $\lambda(\cdot)$ denotes the extremal length and $\|du\|^2$ the Dirichlet integral. This result was essentially known to Ahlfors and Beurling by the time of their fundamental paper on conformal invariants [1]. We observe that if W is planar and α_0, α are each single boundary components, $\exp 2\pi(u+iu^*)/\|du\|^2$ is a conformal mapping of W into $1 < |z| < \exp 2\pi/\|du\|^2$ and the images of the components in β are circular slits and the images of the components in γ radial slits.

The purpose of this paper is to give a complete generalization of the above result to arbitrary open Riemann surfaces. As a consequence of our work we obtain a new class of conformal mappings of plane regions onto "extremal" slit annuli analogous to the situation described above.

We begin with an open Riemann surface W and partition its ideal boundary into four disjoint sets $\alpha_0, \alpha, \beta, \gamma$ with α_0 and α non-empty and α_0, α and $\alpha_0 \cup \alpha \cup \beta$ closed in the Kerékjárto-Stoilöw compactification $\hat{W}$ of W. Classes of curves $\mathcal{F}$, $\mathcal{F}^*$ analogous to F and F^*

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