CURVES ON 2-MANIFOLDS: A COUNTEREXAMPLE

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In [1] R. Baer proved that if two simple closed curves on a closed orientable 2-manifold M of genus greater than one are homotopic, but not homotopic to zero, then they are isotopic. It is well-known that this theorem is true without restriction on M (see [2] for example). One might be tempted to assert the stronger: if two simple closed curves are homotopic keeping a basepoint fixed, then they are isotopic keeping the basepoint fixed. In this paper we show that the stronger result is not true in general. The counterexample has the property that each simple closed curve is the boundary of a Möbius band in M. In [2] it is proved that this is the only type of counterexample.

If $f: X \times I \to Y \times I$ is a level preserving imbedding, we say that $f_0, f_1: X \to Y$ are *isotopic*.

THEOREM 1. *Let $f_0, f_1: S^1, * \to M, *$ be imbeddings of simple closed curves which bound disks with opposite orientations. (Note that we can define orientations in a neighbourhood of the basepoint, even if M is non-orientable.) Then f_0 is isotopic to f_1 if and only if M is a 2-sphere.*

Proof. Suppose $M \neq S^2$ and f_0 is isotopic to f_1. We shall deduce a contradiction. If M is non-orientable, let M' be the orientable double cover and let $f_0', f_1': S^1, * \to M', *$ be liftings of f_0, f_1. Let $\tau: M' \to M'$ be the covering translation. An isotopy between f_0 and f_1, keeping the basepoint fixed, can be lifted to an isotopy between f_0' and f_1' in $M' - \tau *$. Since $M' - \tau * \neq S^2$, there is no loss of generality in assuming that M is orientable.

Let $F_t: S^1, * \to M, *$ be the isotopy between f_0 and f_1. Since $f_t \simeq 0$, $f_t S^1$ bounds a disk D_t. Since $M \neq S^2$, $f_t S^1$ bounds only one disk. $f_t S^1$ assigns an orientation to D_t and hence to M for each t. It is easy to see that this orientation is unchanged by a small change in t. (For example remove a point p from $\operatorname{int} D_t$ and a point q from $M - D_t$. Then f_t gives a homology class in $H_1(M - p - q)$, which determines the orientation of M.) It follows that $f_0 S^1$ and $f_1 S^1$ bound disks with the same orientation, which contradicts our hypothesis.