LINEAR EXTREMAL PROBLEMS FOR ANALYTIC FUNCTIONS

BY

DENNIS A. HEJHAL

Stanford University, Stanford, Calif., U.S.A. (1), (2)

I. Introduction

In recent years the classical Schwarz lemma has been generalized in at least two important ways. The first way is the following. Let D be any plane domain. Let ξ be any point in D. Denote by $\mathcal{C}$ the family of all single-valued analytic functions $f(z)$ on D such that $|f(z)|\leqslant 1$ and $f(\xi)=0$. Define $M=\sup\{|f'(\xi)|; f\in\mathcal{C}\}$. Since $\mathcal{C}$ is a nonvoid normal family, there will be at least one F in $\mathcal{C}$ such that $F'(\xi)=M$. The problem is to study such extremal functions F. Generally speaking, the basic work on this problem when D is of finite connectivity was done by L. Ahlfors [1, 2], P. Garabedian [7], and H. Grunsky [9, 10, 11, 12]. Ahlfors and Grunsky proved results about the uniqueness and the boundary behavior of the extremal function. Garabedian treated the problem as a so-called dual extremal problem and expressed the extremal function in terms of the Szegö kernel function. The case in which D is an arbitrary plane domain (possibly of infinite connectivity) was first studied in detail around 1960 by L. Carleson [5] pp. 73–82 and S. Ja. Havinson [14]. Both of these authors established the uniqueness of the extremal function and Havinson went on to discuss the behavior of the extremal function in great depth.

The second type of generalization consists essentially of replacing the expression $f'(\xi)$ above by an arbitrary linear functional $\mathcal{L}[f]$ and by replacing the auxiliary condition $f(\xi)=0$ by a more general restriction on the zeros of $f(z)$. For an obvious reason, the problem is now called a linear extremal problem. In the case of finite connectivity, such linear extremal problems have been studied, for example, by S. Ja. Havinson [13] and P. Lax [20].

Thus far, a detailed study of the corresponding general linear extremal problems on arbitrary plane domains—not to mention open Riemann surfaces—seems to be lacking.

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