VECTOR FIELDS WITH FINITE SINGULARITIES

BY

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I. Introduction

In this paper we give some generalizations of the famous theorem of H. Hopf which states that the number of singularities of a tangent vector field on a compact smooth manifold is equal to the Euler characteristic. Instead of a single vector field we consider r vector fields $u_1, \dots, u_r$ and we are interested in their "singularities", that is, the set Σ of points on the manifold at which they become linearly dependent. In general Σ will have dimension $r-1$, it is a cycle(1) and its homology class is the $(n-r+1)$th Stiefel-Whitney class of the manifold. This is the standard primary obstruction theory and it provides one way of generalizing the classical Hopf Theorem. However, this theory says nothing about Σ if $\dim \Sigma < r-1$. In this paper following E. Thomas [20] we shall generalize the Hopf theorem by considering the other extreme case in which Σ is finite, so that $\dim \Sigma = 0$. General homotopy theory tells us that we are now involved in higher order obstruction theory and that the situation is much more complicated, as we shall now explain.

For each point $A \in \Sigma$ we have a local obstruction (2)

$$O_A(u_1, \dots, u_r) \in \pi_{n-1}(V_{n,r})$$

where $V_{n,r} = SO(n)/SO(n-r)$ is the Stiefel manifold of orthogonal r-frames in $\mathbf{R}^n$. In local coordinates $(x_1, \dots, x_n)$ with origin A, O_A is just the homotopy class of the map of a small sphere $\sum x_i^2 = \varepsilon$ into (3) $W_{n,r} = GL(n, \mathbf{R})/GL(n-r, \mathbf{R})$ given by $x \mapsto u_1(x), \dots, u_r(x)$. The vanishing of O_A is the necessary and sufficient condition that we can deform $u_1, \dots, u_r$

(1) With integer or mod 2 coefficients depending on the parity of r.

(2) Thomas calls O_A the index at A. Since our methods involve using the index theory of elliptic operators we prefer a different terminology.

(3) As is well-known $V_{n,r} \to W_{n,r}$ is a homotopy equivalence (equivalently every r-frame can be naturally orthogonalized).