ON SYSTEMS OF IMPRIMITIVITY ON LOCALLY COMPACT ABELIAN GROUPS WITH DENSE ACTIONS

BY

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1. Introduction

Let Γ be a countable dense subgroup of the group R of real numbers with usual topology. Give Γ the discrete topology and let $B=\hat{\Gamma}$ be its compact dual. For each $t \in R$, the function $\exp(it\lambda)$, $\lambda \in \Gamma$, is a character on Γ, which we denote by e_t. Then the map, $\varphi: t \to e_t$, is a continuous isomorphism of R into B and $\varphi(R)$ is dense in B. We assume that $2\pi \in \Gamma$. Let K denote the annihilator of the subgroup Γ_0 generated by 2π. The group $N=K \cap \varphi(R)$ consists of elements $\{e_n\}$, $n=0, \pm 1, \pm 2, \ldots$ and it is dense in K. In [3] Gamelin showed that every (N, K) cocycle gives rise, in a natural way, to an (R, B) cocycle, and that in any cohomology class of (R, B) cocycles there is a cocycle obtained from an (N, K) cocycle by his procedure. Gamelin considered only scalar cocycles. As a consequence of this work he was able to resolve some of the problems raised by Helson in [5 (1965)] on compact groups with ordered duals.

If a subgroup G_0 of a locally compact group G acts on G through translation, then by (G_0, G) system of imprimitivity we mean a system of imprimitivity for G_0 based on G, acting in some separable Hilbert space $\mathcal{H}$. In this paper we show that each (N, K) system of imprimitivity (V, E) gives rise to an (R, B) system of imprimitivity $(\bar{V}, \bar{E})$. If U denotes the unitary group (indexed by $\hat{K}=\Gamma/\Gamma_0$) associated with E, and F denotes the spectral measure of V (defined on Borel subsets of T, the circle group), then (U, F) is a $(\hat{K}, T)$ system of imprimitivity. We show that (U, F) gives rise in a natural way to a (Γ, R) system of imprimitivity $(\tilde{U}, \tilde{F})$, and that every (Γ, R) system of imprimitivity is equivalent to a system of imprimitivity $(\tilde{U}, \tilde{F})$. Finally if $\bar{U}$ denotes the unitary group indexed by Γ with spectral measure $\bar{E}$ and $\bar{F}$ the spectral measure of $\bar{V}$, then $(\bar{U}, \bar{F})$ and $(\tilde{U}, \tilde{F})$ are equivalent systems of imprimitivity. We thus complete the circle of ideas involved in Gamelin's work.

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