# INTEGRAL MEANS, UNIVALENT FUNCTIONS AND CIRCULAR SYMMETRIZATION

BY

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## 1. Introduction

We begin by considering the class $S$ of all functions $f(z)$ holomorphic and univalent in the unit disk $|z|<1$ with $f(0)=0$, $f'(0)=1$, and denote by $k(z)$ the Koebe function,

$$k(z)=\frac{z}{(1-z)^2},$$

which maps the unit disk conformally onto the complex plane slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. The Koebe function is known to be extremal for many problems involving $S$. The first result in this paper asserts this is the case for a large class of problems about integral means. Specifically, I will prove the following theorem.

THEOREM 1. *Let $\Phi$ be a convex non-decreasing function on $(-\infty, \infty)$. Then for $f\in S$ and $0<r<1$,*

$$\int_{-\pi}^{\pi}\Phi(\log|f(re^{i\theta})|)\,d\theta\leqslant\int_{-\pi}^{\pi}\Phi(\log|k(re^{i\theta})|)\,d\theta. \tag{1}$$

*If equality holds for some $r\in(0,1)$ and some strictly convex $\Phi$, then $f(z)=e^{-i\alpha}k(ze^{i\alpha})$ for some real $\alpha$.*

In particular, we have for $0<r<1$,

$$\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta\leqslant\int_{-\pi}^{\pi}|k(re^{i\theta})|^p\,d\theta\quad(0<p<\infty),$$

$$\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta\leqslant\int_{-\pi}^{\pi}\log^+|k(re^{i\theta})|\,d\theta. \tag{2}$$

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