K-THEORY OBSTRUCTIONS TO THE EXISTENCE OF VECTOR FIELDS

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1. Introduction

In the paper [7] with M. F. Atiyah, we showed how to apply K-theory for computing the top dimensional obstruction to the existence of r linearly independent vector fields on an oriented manifold X. The purpose of this note is to extend the method of [7] to apply also for the other higher obstructions.

Following classical obstruction theory as developed for example in Steenrod's book [16, part 3] we fix a triangulation of the n-dimensional oriented closed manifold X, and construct the vector fields successively over the q-skeleton X^q. Assume that the set $\mathbf{u} = \{u_1, ..., u_r\}$ is defined and linearly independent over X^{q-1}. As is well-known this gives rise to a natural obstruction cocycle

$$\mathfrak{o}(\mathbf{u}) \in C^q(X, (\pi_{q-1}(V_{n,r}))^t)$$

in the cochain complex of X with coefficients in the local coefficient system which restricted to a q-simplex σ^q of X is the $(q-1)$-th homotopy group of the Stiefel manifold of r-frames in the tangent space at the 1-st vertex of σ^q. As X is assumed to be oriented this coefficient system is actually trivial. The cohomology class

$$\{\mathfrak{o}(\mathbf{u})\} \in H^q(X, \pi_{q-1}(V_{n,r}))$$

is the obstruction to deforming $\mathbf{u}$ (relative to X^{q-2}) into a set which has an extension over X^q. As an example of our results we shall prove the following theorem.

THEOREM 1.1. *Let X be a manifold as above of dimension $n = 4k - s \geq 6$, and let $\mathbf{u} = \{u_1, u_2, u_3\}$ be three linearly independent vector fields over X^{n-2}. Then for $s \neq 3$ we have $\{\mathfrak{o}(\mathbf{u})\} = 0$ in $H^{n-1}(X, \pi_{n-2}(V_{n,3}))$.*

If $s = 3$ then $\pi_{n-2}(V_{n,3}) = \mathbf{Z}/4$, and assuming $H_1(X, \mathbf{Z})$ has no 2-torsion we have