ZEROS OF THE DERIVATIVES OF THE RIEMANN ZETA-FUNCTION

BY

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1. Introduction and some results

Several diverse theorems concerning the zeros of $\zeta^{(k)}(s)$, the kth derivative, of the Riemann zeta function will be presented. Relationships with existing results, [1], [5–9], will be discussed.

THEOREM 1. *Let $N^-(T)$ be the number of zeros of $\zeta(s)$ in R: $0<t<T$, $0<\sigma<\frac{1}{2}$ where $s=\sigma+it$. Let $N_1^-(T)$ be the number of zeros of $\zeta'(s)$ in R. Then*

$$N_1^-(T) = N^-(T) + O(\log T). \tag{1.1}$$

Unless $N^-(T) > T/2$ for all large T there exists a sequence $\{T_j\}$, $T_j \to \infty$ as $j \to \infty$ such that

$$N_1^-(T_j) = N^-(T_j). \tag{1.2}$$

Theorem 1 can be regarded as stating that $\zeta(s)$ and $\zeta'(s)$ have the same number of zeros in $0<\sigma<\frac{1}{2}$. The following is essentially due to Speiser [5].

COROLLARY TO THEOREM 1. *The Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0<\sigma<\frac{1}{2}$.*

One half of the above, namely RH $=>$ $\zeta'(s)$ is zero-free in $0<\sigma<\frac{1}{2}$ was rediscovered by Spira [9].

Let $N_k(T)$ be the number of non-real zeros of $\zeta^{(k)}(s)$ for $0<t<T$. Then it was shown by Berndt [1], and will also be a by-product of the proof of Theorem 2, that for $k \geqslant 1$

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