# POLYTOPE PAIRS AND THEIR RELATIONSHIP TO LINEAR PROGRAMMING

BY

VICTOR KLEE

*University of Washington, Seattle, Washington 98195, USA*

## Introduction

As the terms are used here, a *polyhedron* is the intersection of a finite number of closed halfspaces in a finite-dimensional real vector space, a *pointed polyhedron* is one whose vertex set is nonempty, and a *polytope* is a bounded polyhedron; equivalently, a polytope is the convex hull of a finite set of points. Prefixes indicate dimension, and the $(d-1)$-faces of a d-polyhedron are its *facets*. A polyhedron of *class* $(d, n)$ is one that is pointed, $d$-dimensional, and has precisely $n$ facets; necessarily, $n \geqslant d$, with $n > d$ in the case of polytopes. A pointed $d$-polyhedron is *simple* provided that each of its vertices is incident to precisely $d$ edges or, equivalently, to precisely $d$ facets. A polytope is *simplicial* provided that each of its facets is a simplex. For properties of polyhedra and polytopes that are used here without explicit reference, are Grünbaum [10]. In particular, basic properties of the duality or polarity of polytopes are used freely [10, pp. 46–49].

Two landmarks in the theory of polytopes were the proofs that as $P$ ranges over all simple polytopes of class $(d, n)$, the minimum and maximum of $v(P)$ (number of vertices of $P$) are equal respectively to

$$(n-d)(d-1)+2$$

and to

$$\gamma(d, n) = \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d}.$$

These results, due respectively to Barnette [1] and McMullen [22], are here extended to certain pairs consisting of a polytope and one of its facets.

For $3 \leqslant d \leqslant u < n$, a pair $(P, F)$ is called a *polytope pair of class* $(d, n, u)$ provided that $P$ is a simple polytope of class $(d, n)$ and $F$ is a facet intersecting precisely $u$ other facets of $P$; $F$ is then a simple polytope of class $(d-1, u)$. The set of all such pairs is denoted by