THE MINIMUM OF A BILINEAR FORM.

By

E. S. BARNES

of Trinity College, CAMBRIDGE.

1. In this paper I investigate the lower bound $M(B)$ of a bilinear form

$$B(x, y, z, t) = \alpha x z + \beta x t + \gamma y z + \delta y t, \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are real, and x, y, z, t take all integral values subject to

$$x t - y z = \pm 1. \tag{1.2}$$

We say that two bilinear forms are equivalent if one may be transformed into the other by a substitution

$$\begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} x' & z' \\ y' & t' \end{pmatrix} \quad or \quad \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} z' & x' \\ t' & y' \end{pmatrix}, \tag{1.3}$$

where p, q, r, s are integers and $ps - qr = \pm 1$. It is clear that equivalent forms assume the same set of values for integral x, y, z, t subject to (1.2), and so have the same lower bound $M(B)$. Further, if we set

$$\Delta = \Delta(B) = \alpha\delta - \beta\gamma,$$
$$\theta = \theta(B) = |\beta - \gamma|,$$

then Δ and θ are invariants of B under equivalence transformation, of weights two and one respectively.

Associated with a bilinear form B is the quadratic form

$$Q(x, y) = B(x, y, x, y) = \alpha x^2 + (\beta + \gamma) x y + \delta y^2, \tag{1.4}$$

of discriminant

$$D = (\beta + \gamma)^2 - 4\alpha\delta = \theta^2 - 4\Delta. \tag{1.5}$$

If two bilinear forms are equivalent under a transformation (1.3), then, putting $x = z$, $y = t$, we see that the associated quadratic forms are also equivalent. Conversely, a quadratic form $Q(x, y) = a x^2 + b x y + c y^2$ is associated with the two bilinear forms