ON THE EXISTENCE OF CERTAIN SINGULAR INTEGRALS.

By

A. P. CALDERON and A. ZYGMUND

Dedicated to Professor Marcel Riesz, on the occasion of his 65th birthday

Introduction.

Let $f(x)$ and $K(x)$ be two functions integrable over the interval $(-\infty, +\infty)$. It is very well known that their composition

$$\int_{-\infty}^{+\infty} f(t)\, K(x-t)\, dt$$

exists, as an *absolutely convergent integral*, for almost every x. The integral can, however, exist almost everywhere even if K is not absolutely integrable. The most interesting special case is that of $K(x) = 1/x$. Let us set

$$\tilde{f}(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\, dt.$$

The function $\tilde{f}$ is called the conjugate of f (or the Hilbert transform of f). It exists for almost every value of x in the Principal Value sense:

$$\tilde{f}(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} \right) \frac{f(t)}{x-t}\, dt.$$

Moreover it is known (See [9] or [7], p. 317) to satisfy the M. Riesz inequality

$$(1) \qquad \left[\int_{-\infty}^{+\infty} |\tilde{f}|^p\, dx \right]^{1/p} \le A_p \left[\int_{-\infty}^{+\infty} |f|^p\, dx \right]^{1/p}, \qquad 1 < p < \infty,$$

where A_p depends on p only. There are substitute result for $p = 1$ and $p = \infty$. The limit $\tilde{f}$ exists almost everywhere also in the case when $f(t)\, dt$ is replaced there by $dF(t)$, where $F(t)$ is any function of bounded variation over the whole interval $(-\infty, +\infty)$. (For all this, see e.g. [7], Chapters VII and XI, where also bibliographical references can be found).