A NOTE ON THE CAPILLARY PROBLEM

BY

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The purpose of this note is to extend the result of Theorem 3 in the preceding paper [2] to a configuration not amenable to the methods of that reference.

§ 1

Consider an open set $\mathcal{N}$ in Euclidean n-space, bounded in part by a surface Σ of class $C^{(2)}$ whose mean curvature H^{Σ} does not change sign; the sign of H^{Σ} is chosen to be positive when the curvature vector points into $\mathcal{N}$. As in [2], we use the symbols Σ and $\mathcal{N}$ to denote also the area and volume of these sets.

Let $u(\mathbf{x})$ be a solution in $\mathcal{N}$ of

$$\operatorname{div}\left(\frac{1}{W}\nabla u\right) = nH(\mathbf{x}), \qquad W^2 = 1 + |\nabla u|^2, \tag{1}$$

where $H(\mathbf{x})$ is prescribed in $\mathcal{N}$, and set

$$\mathbf{T}u = \frac{1}{W}\nabla u \tag{2}$$

in $\mathcal{N}$. Denote by $\boldsymbol{\nu}$ the exterior directed unit normal on Σ. We observe first that *if $H(\mathbf{x})$ is bounded on one side, then for any open subset $\Sigma^* \subset \Sigma$, the quantity $\int_{\Sigma^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma$ is uniquely defined as a limit of integrals over surfaces converging in any uniform way to Σ^* from within $\mathcal{N}$*. To see this, it suffices to suppose $H > 0$ and to show the result when Σ^* is the part of Σ lying interior to a (small) sphere S centered on Σ, so that Σ^* and a part $S^* \subset S$ together bound $\mathcal{N}^* \subset \mathcal{N}$. Integrating (1) by parts in the subregion cut off by an approximating surface Γ^*, and passing to the limit as $\Gamma^* \to \Sigma^*$, we find

$$\lim_{\Gamma^* \to \Sigma^*} \int_{\Gamma^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma = \int_{N^*} H(\mathbf{x})\, d\mathbf{x} - \int_{S^*} \mathbf{T}u \cdot \boldsymbol{\nu}\, d\sigma.$$

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