ANOTHER PROOF OF THE EXISTENCE OF SPECIAL DIVISORS

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In the classical literature on curves, it is repeatedly asserted that the divisors of degree n and (projective) dimension at least r depend on at least $(\tau + r)$ parameters, with $\tau = (r+1)(n-r) - rg$ where g is the genus. However, it was only recently proved that, if τ is nonnegative, such divisors exist. Meis [17] gave an analytic proof for the case $r=1$, which involves deforming the curve over the Teichmüller space into a special curve that has an explicit $(\tau + r)$-dimensional family of such divisors. Kempf [10] and Kleiman-Laksov [12] independently gave similar algebraic proofs for the case of any r, valid in any characteristic, which involve constructing a global, finite cohomology complex for the Poincaré sheaf and performing a local analysis of some resulting determinantal subvarieties of the jacobian. Gunning [7], working over the complex numbers, gave a proof for the case $r=1$, which uses Macdonald's description of the homology of the symmetric product and Hensel-Landsberg's trick of choosing n minimal ([8], lecture 31, § 3, p. 550); otherwise, the proof is akin to the proofs in [10] and [12].

We offer below another algebraic proof for the general case, which is conceptually simpler and more natural. Our framework is the theory of singularities of mappings. We observe that the divisors of degree n and dimension at least r are parametrized by the scheme Z_r of first order singularities of rank at least r of the canonical map from the variety of divisors of degree n to the jacobian, a base point on the curve having been picked. (By definition, Z_r consists of the points where the map of tangent sheaves is of rank at least r.) Consequently, if Z_r is empty or of the right dimension, then in the Chow ring its class is given by a certain Thom polynomial in the Chern classes of the variety of divisors. The Thom polynomial is evaluated (Proposition 16), and the value is then proved nonzero if τ is nonnegative by using the projection formula and Poincaré's formulas. Hence, divisors of degree n and dimension at least r must exist; otherwise, the value of the Thom polynomial would be zero. In fact they abound: carried further, the proof shows that,