THE TOPOLOGY OF SPACES OF RATIONAL FUNCTIONS

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§ 1. Introduction

A rational function f of the form

$$f(z)=\frac{p(z)}{q(z)}=\frac{z^n+a_1z^{n-1}+\ldots+a_n}{z^n+b_1z^{n-1}+\ldots+b_n}, \tag{1}$$

where a_i and b_i are complex numbers, defines a continuous map of degree n from the Riemann sphere $S^2=\mathbf{C}\cup\infty$ to itself. If the coefficients $(a_1, ..., a_n; b_1, ..., b_n)$ vary continuously in $\mathbf{C}^{2n}$ the map f varies continuously providing the polynomials p and q have no root in common; but the topological degree of the map f jumps when a root of p moves into coincidence with a root of q.

Let F_n^* denote the open set of $\mathbf{C}^{2n}$ consisting of pairs of monic polynomials (p, q) of degree n with no common root. F_n^* is the complement of an algebraic hypersurface, the "resultant locus", in $\mathbf{C}^{2n}$. On the other hand it can be identified with a subspace of the space M_n^* of maps $S^2\to S^2$ which take ∞ to 1 and have degree n. In this paper I shall prove that when n is large the $2n$-dimensional complex variety F_n^* is a good approximation to the homotopy type of the space M_n^*, or, more precisely

PROPOSITION (1.1). *The inclusion $F_n^*\to M_n^*$ is a homotopy equivalence up to dimension n.*

Equivalently one can consider the space F_n of rational functions of the form

$$\frac{a_0z^n+\ldots+a_n}{b_0z^n+\ldots+b_n},$$

where again the numerator and denominator have no common factor, and a_0 and b_0 are not both zero. This space is the complement of a hypersurface in $\mathbf{P}^{2n+1}$. It can be regarded as a subspace of the space M_n of all maps $S^2\to S^2$ of degree n. Proposition (1.1) implies at once