AUTOMORPHISMS AND INVARIANT STATES OF OPERATOR ALGEBRAS

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1. Introduction

Let $\mathfrak{A}$ be a von Neumann algebra and G a group of *-automorphisms of $\mathfrak{A}$ with fixed point algebra $\mathcal{B}$ in $\mathfrak{A}$. If $\mathfrak{A}$ is semi-finite and $\mathcal{B}$ contains the center of $\mathfrak{A}$ the normal G-invariant states of $\mathfrak{A}$ were analysed in [3], [12], [13]. In the present paper we shall extend these studies to the general situation, in which the center is not necessarily left fixed by G. The main result, from which the rest follows, states that if $\mathfrak{A}$ is semi-finite and ω a faithful normal G-invariant state of $\mathfrak{A}$, and if G acts ergodicly on the center of $\mathfrak{A}$, then there exists a faithful normal G-invariant semi-finite trace τ of $\mathfrak{A}$ which is unique up to a scalar multiple, and a positive self-adjoint operator $B \in L^1(\mathfrak{A}, \tau)$ affiliated with $\mathcal{B}$ such that $\omega(A) = \tau(BA)$ for all $A \in \mathfrak{A}$. For example, if G is ergodic on $\mathfrak{A}$ then ω is a trace, hence $\mathfrak{A}$ is finite. As an application to C^*-algebras we show that if $\mathcal{A}$ is an asymptotically abelian C^*-algebra (more specifically G-abelian) and ϱ is an extremal G-invariant state, then either the weak closure of its representation, viz $\pi_\varrho(\mathcal{A})''$, is of type III, or the cyclic vector x_ϱ such that $\varrho(A) = (\pi_\varrho(A)x_\varrho, x_\varrho)$, $A \in \mathcal{A}$, is a trace vector for the commutant of $\pi_\varrho(\mathcal{A})$. This has previously been shown for invariant factor states [12].

The basic technical tool used in this paper is the theory of Tomita [15] and Takesaki [14] on the modular automorphisms associated with faithful normal states of von Neumann algebras. It will, however, mainly be applied to semi-finite algebras. We recall from [14] that if $\mathfrak{A}$ is a von Neumann algebra with a separating and cyclic vector x_0 then the *-operation S: $Ax_0 \to A^*x_0$ is a pre-closed conjugate linear operator with polar decomposition $S = J\Delta^{\frac{1}{2}}$, where J is a conjugation of the underlying Hilbert space, and Δ is a positive self-adjoint operator—the modular operator defined by x_0. The modular automorphism σ_t of $\mathfrak{A}$ associated with x_0 (or rather the state ω_{x_0}) is given by $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$. Furthermore, J satisfies the relation $J\mathfrak{A}J = \mathfrak{A}'$. For details and further results from this

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