RANDOM DIFFERENCE EQUATIONS AND RENEWAL THEORY FOR PRODUCTS OF RANDOM MATRICES

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Introduction

In this paper we study the limit distribution of the solution Y_n of the difference equation

$$Y_n = M_n Y_{n-1} + Q_n, \quad n \geqslant 1, \tag{1.1}$$

where M_n and Q_n are random $d \times d$ matrices respectively d-vectors and Y_n also is a d-vector. Throughout we take the sequence of pairs (M_n, Q_n), $n \geqslant 1$, independently and identically distributed. The equation (1.1) arises in various contexts. We first met a special case in a paper by Solomon, [20] sect. 4, which studies random walks in random environments. Closely related is the fact that if $Y_n(i)$ is the expected number of particles of type i in the nth generation of a d-type branching process in a random environment with immigration, then $Y_n = (Y_n(1), ..., Y_n(d))$ satisfies (1.1) (Q_n represents the immigrants in the nth generation). (1.1) has been used for the amount of radioactive material in a compartment ([17]) and in control theory [9a]. Moreover, it is the principal feacture in a model for evolution and cultural inheritance by Cavalli-Sforza and Feldman [2]. Notice also that the dth order linear difference equation

$$y_n = \sigma_n^{(1)} y_{n-1} + \sigma_n^{(2)} y_{n-2} \dots + \sigma_n^{(d)} y_{n-d} + q_n$$

can be brought into the form (1.1), if one takes

$$Y_n = (y_{n+d-1}, y_{n+d-2}, ..., y_n), \quad Q_n = (q_{n+d-1}, 0, ..., 0)$$

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