COHOMOLOGY OF OPERATOR ALGEBRAS

I. *Type I von Neumann algebras*

BY

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1. Introduction

The cohomology theory of associative linear algebras over an arbitrary field was initiated and developed by Hochschild [5, 6, 7]. With $\mathfrak{A}$ an algebra and $\mathcal{M}$ a two-sided $\mathfrak{A}$-module, the linear space $C^n(\mathfrak{A}, \mathcal{M})$ of *n-cochains* consists of all n-linear mappings from $\mathfrak{A} \times \mathfrak{A} \times \ldots \times \mathfrak{A}$ into $\mathcal{M}$. The *coboundary operator* Δ maps $C^n(\mathfrak{A}, \mathcal{M})$ linearly into $C^{n+1}(\mathfrak{A}, \mathcal{M})$ for each $n = 0, 1, 2, \ldots$, and satisfies $\Delta^2 = 0$. With $Z^n(\mathfrak{A}, \mathcal{M})$ the null-space of Δ in $C^n(\mathfrak{A}, \mathcal{M})$, and $B^{n+1}(\mathfrak{A}, \mathcal{M})$ the range of Δ in $C^{n+1}(\mathfrak{A}, \mathcal{M})$, we have $B^n(\mathfrak{A}, \mathcal{M}) \subseteq Z^n(\mathfrak{A}, \mathcal{M})$ $(n = 1, 2, \ldots)$. The quotient space $Z^n(\mathfrak{A}, \mathcal{M})$ is called the *n-dimensional cohomology group* of $\mathfrak{A}$, with coefficients in $\mathcal{M}$, and is denoted by $H^n(\mathfrak{A}, \mathcal{M})$.

The present paper is concerned with cohomology groups of operator algebras. For such algebras there are several possible cohomology theories, closely analogous to the Hochschild theory in algebraic structure, but differing from one another in the nature and extent of the topological properties required of the module $\mathcal{M}$, the action of $\mathfrak{A}$ on $\mathcal{M}$, and the n-linear mappings which are admitted as n-cochains. The Hochschild theory itself is available but, with one important exception, the problem of computing the (purely algebraic) cohomology groups $H^n(\mathfrak{A}, \mathcal{M})$, with $\mathfrak{A}$ a C^*-algebra and $\mathcal{M}$ a two-sided $\mathfrak{A}$-module, seems intractable. The exceptional case, which has provided much of the motivation for the work in this paper, arises from the fact that a von Neumann algebra $\mathcal{R}$ has no outer derivations ([10, 13]; for a later proof see [9]). This result can easily be reformulated, in terms of the Hochschild cohomology theory, as the assertion that $H^1(\mathcal{R}, \mathcal{R}) = 0$. A number of other problems concerning derivations of operator algebras can be expressed in cohomological terms. The present paper treats the case in which the module $\mathcal{M}$ is the dual space of some Banach space, and the bilinear mappings $(A, m) \to Am$ and $(A, m) \to mA$ (from $\mathfrak{A} \times \mathcal{M}$ into