THE MAXIMUM PRINCIPLE FOR MULTIPLE-VALUED ANALYTIC FUNCTIONS

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I. Introduction

If F is a single-valued analytic function satisfying $|F(z)| \leqslant 1$ throughout a domain Ω in the Riemann sphere, then of course $|F(\zeta)| \leqslant 1$ for any particular ζ. We have $|F(\zeta)| = 1$ only if F is a constant of absolute value one. The same statements hold even if F is not necessarily single-valued but has single-valued absolute value, for $\log |F|$ is still subharmonic. In particular if F is not single-valued then

$$\limsup_{z \to \partial\Omega} |F(z)| \leqslant 1$$

implies the strict inequality $|F(\zeta)| < 1$. Among the concerns of the present paper is the question of how small $|F(\zeta)|$ must be, given that F has a particular type of multiple-valued behavior.

This multiple-valued behavior may be abstracted in the following way as a character (homomorphism into the group T of complex numbers of absolute value 1) of the fundamental group of Ω. Continuation of a function element of F along a cycle γ results in multiplication by a constant of absolute value 1, which we call $\Gamma_F(\gamma)$. This constant is easily seen to be independent both of the starting point on γ and the particular element of F chosen. We may write concisely

$$\Gamma_F(\gamma) = \exp\{i \underset{\gamma}{\Delta} \arg F\}.$$

Since homotopic curves produce identical analytic continuations, Γ_F is constant on each homotopy class and may therefore be considered a function on $\pi(\Omega)$, the fundamental group of Ω. It is trivially a character.

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