GRASSMANN ANGLES OF CONVEX POLYTOPES

BY

BRANKO GRÜNBAUM

University of Washington, Seattle, Wash., U.S.A. (1)

Dedicated to Hugo Hadwiger on his 60th anniversary, December 23, 1968

1. Introduction

The aim of the present note is to generalize to quantities we call Grassmann angles the results of Perles–Shephard [6] and Shephard [9] concerning angles and deficiencies of convex polytopes.

We shall define, for each d-polytope $(=d$-dimensional convex polytope) P^d and for each m, $1 \leqslant m \leqslant d-1$, a $(d-m)$-dimensional vector $\gamma^m(P^d) = (\gamma_0^m(P^d), \ldots, \gamma_{d-1-m}^m(P^d))$ related to the shape of P^d at its different faces. It will turn out that the *Grassmann angle sums* $\gamma_j^m(P^d)$ satisfy equations similar to those of Euler and Dehn–Sommerville, and that they satisfy certain inequalities. For $m=1$, $m=2$, or $m=d-1$, the Grassmann angles are related to the usually considered angles, deficiencies, or exterior angles, and our results therefore specialize to known facts concerning those entities.

As a preliminary stage we investigate the corresponding Grassmann angles for convex polyhedral cones; the results obtained contain as special case the theorem of Sommerville [10] relating (for even d) the volume of a spherical d-polytope to its angles, as well as a result of Fáry [2].

The main tool used, which was similarly applied in special cases already in Fáry [2], Perles–Shephard [6], and Shephard [9], is the invariant measure on the Grassmann manifold of all m-flats through the origin of the Euclidean d-space E^d. The measure-theoretic approach permits to restrict the consideration to m-flats which are in "general position" with respect to the polytope or cone considered, thus eliminating the necessity to take into account various "singular situations".

We start by investigating the Grassmann angles of convex cones (Section 2); in Section 3 we consider the Grassmann angles of polytopes, while the concluding Section 4 is devoted to some additional remarks and problems.

We shall freely use the standard results on convex polytopes; facts for which no references are given may be found in [4]; an account of the older results on angle-sums and their history is also given in [4].

(1) Research supported in part by Office of Naval Research contract N00014–67–A–0103–0003.