NON-UNITARY DUAL SPACES OF GROUPS

BY

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1. Introduction

The infinite-dimensional unitary representations of an arbitrary locally compact group G have been extensively studied since 1947. For some purposes, however, the unitary restriction is very undesirable—for example, if we wish to carry out "analytic continuation" of representations of G. This paper investigates some general concepts concerning non-unitary representations. Extending the ideas of [3], we define a "non-unitary dual space" $\hat{G}$ of G. Roughly speaking, $\hat{G}$ is the space of all equivalence classes of irreducible (not necessarily either unitary or finite-dimensional) representations of G. It is not however a trivial matter to decide what we ought to mean by 'representation', 'irreducible', or 'equivalence class'. At first sight it might appear reasonable to restrict ourselves to representations living in a Banach space. We shall therefore begin with an example showing that Banach spaces form too narrow a framework if we have in mind analytic continuation of representations of general groups.

Let G be the Galilean group, that is, the three-dimensional nilpotent Lie group of all triples of real numbers, multiplication being given by $\langle a,b,c\rangle\langle a',b',c'\rangle=\langle a+a',b+b',c+c'-ab'\rangle$. The unitary representations of G are well known (see [15]). For each non-zero real number λ there is a unique (infinite-dimensional) irreducible unitary representation T^{λ} of G with the property that, for each real c, T^{λ} sends the central element $\langle 0,0,c\rangle$ of G into the scalar operator $e^{i\lambda c}\cdot 1$. One would hope by a process of "analytic continuation" to obtain non-unitary irreducible representations T^{λ} having the same property for *complex* λ. But we shall now show that such a T^{λ} could not live in a Banach space. Indeed: Let us write $\gamma_1(a)=\langle a,0,0\rangle$, $\gamma_2(b)=\langle 0,b,0\rangle$, $\gamma_3(c)=\langle 0,0,c\rangle$ (a,b,c real); and let us suppose that T is a homomorphism of G into the group of bounded invertible operators on some Banach space H such that $T_{\gamma_3(c)}=e^{i\lambda c}\cdot 1$ for all real c, where λ is a non-real complex number (and 1 is the identity operator on H). Since $\gamma_1(-1)\gamma_2(b)\gamma_1(1)=\gamma_3(b)\gamma_2(b)$, we have