HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM–LIOUVILLE FUNCTIONS

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1. Introduction

An examination of tables [3; 5; 8; 12; 15] of the positive(2) zeros of familiar special functions, such as Bessel functions(3) and various orthogonal polynomials, suggests that sequences of differences constructed from those zeros behave in a regular manner. Indeed, certain heuristically observed regularities are exploited systematically by table-makers as checks on their computations [1; 5, p. 404; 12, esp. pp. liii–liv].(4)

Rigorous study of this useful phenomenon, however, does not appear to have progressed beyond consideration of the second differences of zeros of Sturm–Liouville functions (solutions of the Sturm–Liouville differential equation $y''+f(x)y=0$). Here Sturm's comparison theorem [13; 14, pp. 19–21] has been the principal tool.

For instance, denoting by $\{c_{\nu n}\}$, $n=1,2,\ldots$, the ascending sequence of positive zeros of an arbitrary Bessel function $\mathcal{C}_\nu(x)$ of order ν, Ch. Sturm [13, pp. 173–175] used his comparison theorem to show that the second (forward) differences $\Delta^2 c_{\nu n}$, $n=1,2,\ldots$, are all positive if $|\nu|<\frac{1}{2}$ and are all negative if $|\nu|>\frac{1}{2}$. In the same manner, similar results have been established for Hermite, Laguerre and Legendre polynomials and other Sturm–Liouville unctions.

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(2) All quantities discussed throughout this paper are assumed to be real.

(3) By a Bessel function we mean *any* real solution of the Bessel differential equation, not merely J_ν or Y_ν.

(4) The regularities now used to check tables are not the ones discussed in this paper. However, the ones established here can also be used conveniently for this purpose.