ON MAHLER'S CLASSIFICATION OF TRANSCENDENTAL NUMBERS

BY

A. BAKER

Trinity College, Cambridge

1. Introduction

According to the well-known classification introduced by Mahler [5] in 1932, the transcendental numbers are divided into three disjoint classes, termed the S-numbers, T-numbers and U-numbers, depending upon which of three possible conditions of approximation the numbers satisfy. A full account of this classification is given in Schneider [11] (Kap. 3) and we refer there for the details. An important feature of the classification is that algebraically dependent numbers belong to the same class. Further subdivisions of the classes have been given, the S-numbers having been classified according to "type" (see [11], p. 67), and the U-numbers according to their "degree" (see [3]). The existence of U-numbers of each degree was proved by LeVeque [3], but it is not known whether there are any T-numbers, or even S-numbers of type exceeding 1.

It is the main purpose of the present paper to investigate how Mahler's classification for real transcendental numbers is related to the more direct classification in which the numbers are divided into two sets according as the regular continued fraction has bounded or unbounded partial quotients. We show that, in fact, there is little correlation; both sets of real numbers, those with bounded partial quotients and those with unbounded partial quotients, contain U-numbers, and also either T-numbers or S-numbers of arbitrarily high type. It follows, incidentally, that at least one of the two sets, the T-numbers or the S-numbers of type exceeding 1, is not empty.

In order to obtain the results referred to above we first prove a general theorem concerning the approximation of transcendental numbers by numbers in a fixed algebraic number field. This extends a theorem of LeVeque [4] (Ch. 4) which itself is a generalisation of Roth's Theorem [9]. Let K be an algebraic number field and