CUTS

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1. Introduction

What does it mean to cut a topological space X along a subset A? Consider two examples:

(1) X is the plane, and A is a triod (i.e. a Y).

(2) X is a Möbius band, and A is the equator.

Note that both sets A have empty interior, or, in the terminology of [20], are *thin*; this is necessary if "cutting" is to make much sense. Now in both examples, it is intuitively clear what happens when X is cut along A: The space X is replaced by a space $\mathbf{X}$, and if (2) $p: \mathbf{X} \twoheadrightarrow X$ is the function which maps each point of $\mathbf{X}$ to the point of X where it came from before cutting, while $\mathbf{A}$ denotes $p^{-1}(A)$, then p maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$. In (1), $\mathbf{X}$ is the plane with a (topologically) circular hole, and $\mathbf{A}$ is the boundary of the hole. In (2), where cutting is occasionally performed as a parlor trick, $\mathbf{X}$ is a cylinder, $\mathbf{A}$ is a circle which is one of the two components of the boundary of $\mathbf{X}$, and $p|\mathbf{A}$ is a double covering. Let us try to identify those common properties of p and $\mathbf{A} \subset \mathbf{X}$ which will lead to a general concept of cutting. (3)

First of all, p is continuous and closed, and—as observed above—maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$. Moreover, in both examples $p|\mathbf{A}$ is finite-to-one, but in general this requirement must be somewhat relaxed, as the following example shows:

(3) In the plane, $\mathbf{X}$ consists of the intervals joining $(0,0)$ to $(x,1)$ for all x in $S=\{1,\frac{1}{2},\frac{1}{3},\ldots,0\}$, and $A=\{(0,0)\}$.

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(2) We use $\twoheadrightarrow$ to denote an *onto* map.

(3) It should be noted that a somewhat different method of cutting was implicitly considered by R. H. Fox in [7]. In many important cases (including Examples (1), (2), and (3)), Fox's cuts agree with ours; in general, however, Fox's map p_F is a restriction of our map p, and the range of p_F (unlike the range of p) need not be all of X. The exact relation between these two ways of cutting will be established in section 16.