On properties of functions with conditions on their mean oscillation over cubes

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Introduction

We consider spaces $BMO_{\varphi,p}$ of functions defined using mean oscillation over cubes in $\mathbf{R}^n$, which include the Morrey spaces $L^{(p,\lambda)}$, the John—Nirenberg space BMO and the Lipschitz spaces Λ_α. It is our purpose to give equivalent characterizations of the spaces $BMO_{\varphi,p}$ and to apply these characterizations to an extension problem for $BMO_{\varphi,p}(G)$, for certain open subsets G of $\mathbf{R}^n$. We prove that the spaces $BMO_{\varphi,p}$ are characterized by a property of the mean oscillation over a class of sets more general than the class of cubes used in their definition.

Although we are mainly interested in the case $1 \leqq p < \infty$, which include BMO $(\varphi(r)=1, p=1)$ and the Morrey spaces, we state some of our results in the more general situation $0<p<\infty$ and $\varphi(r)$ satisfying certain growth conditions. See the remark following Theorem 2.2 in section 2.1.

Let E be a bounded set in $\mathbf{R}^n$ with positive Lebesgue measure. Then the mean oscillation of f over E (in L^p-sense, $0<p\leqq\infty$) is defined by

$$\Omega_p(f, E) = \inf_C \left(|E|^{-1} \int_E |f(x)-C|^p \, dx\right)^{1/p}, \tag{0.1}$$

whenever it is finite. For any such set E there is $C=C_E$ for which the infimum in (0.1) is attained and C_E is from now on defined in this way. Note that C_E is not uniquely defined, cf. J-O. Strömberg [15].

The set function $\Omega_p(f, E)$ is a local best approximation of order zero of f in L^p in the sense of Ju. A. Brudnyi [1, p. 75].

The class of sets we consider is defined by

$$K_r = \{x \in \mathbf{R}^n;\ \text{dist}\ (x, K) \leqq r\},\ r>0, \tag{0.2}$$

where K ranges over all compact sets K with Lebesgue measure zero.