Endomorphisms of finetely generated projective modules over a commutative ring

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Introduction

The origin of this paper is a misprint (?) in Bourbaki ([4], p. 156, Exercise 13 d). There it is stated that if f is a 2×2-matrix with entries in a commutative ring and $f^2 = 0$ then $(\mathrm{Tr}\, f)^4 = 0$ and 4 is the smallest integer with this property. Using the Cayley-Hamilton theorem we get $f^2 - af + b1 = 0$ where $a = \mathrm{Tr}\, f$ and $b = \det f$. Noting that $f^2 = 0$ and taking traces we get $a \cdot \mathrm{Tr}\, f = a^2 = 2b$. Multiplying the first equation by f gives $bf = 0$ which implies $b \cdot \mathrm{Tr}\, f = ba = 0$. Hence $a^3 = 2ab = 0$ so 3 and not 4 is the smallest integer above. Experimenting with small m and n one soon makes the conjecture: If f is an $n\times n$-matrix with $f^{m+1} = 0$ then $(\mathrm{Tr}\, f)^{mn+1} = 0$. This is proved in a somewhat more general setting in 1.7 using exterior algebra.

In Section 1 the characteristic polynomial $\lambda_t(f)$ is defined for an endomorphism $f\colon P \to P$ where P is a finitely generated projective A-module (A is a commutative ring with 1). If P is free then $\lambda_t(f) = \det(1 + tf)$. The exponential trace formula (in case A contains $\mathbf{Q}$)

$$\lambda_t(f) = \exp\left(-\sum_1^\infty \frac{\mathrm{Tr}\,(f^i)}{i}\,(-t)^i\right)$$

connects $\lambda_t(f)$ with the traces of the powers of f.

Various computations of $\lambda_t(f)$ are made in Section 2. By the isomorphism $\mathrm{End}_A(P) \to P^* \otimes_A P$ where $P^* = \mathrm{Hom}_A(P, A)$ every $f\colon P \to P$ corresponds to a tensor $\sum_i x_i^* \otimes x_i$ with $x_i^* \in P^*$, $x_i \in P$. Let $M(f)$ be the matrix with entries $a_{ij} = \langle x_i^*, x_j\rangle$. Then $\lambda_t(f) = \det(1 + tM(f))$. Even the computation of $\lambda_t(1_P)$

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