Intermediate spaces and the class $L \log^+ L$

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1. Introduction

The role in classical analysis of the Orlicz space $L \log^+ L$ is now well-known ([18]; I, pp. 33, 242, 267; II, p. 159) and, quite recently, O'Neil ([14]) has exhibited certain connections with the theory of interpolation of operators. In this note we show how the modern theory of interpolation methods relates to the space $L \log^+ L$ by describing the intermediate spaces $(L^1, L \log^+ L)_{\theta, q; K}$ and $(L \log^+ L, L^\infty)_{\theta, q; K}$ generated by the K-interpolation method of Peetre ([3], Chap. 3). In particular, we find interesting relationships (Corollaries C, E) between these and the Lorentz spaces. In fact the essential observation in the proofs of our results is that all three interpolated spaces L^1, $L \log^+ L$ and L^∞ are Lorentz Λ-spaces ([12]) which enables us ([15]) to identify the functional norm $K(t; f)$. An application of (some variants of) Hardy's inequality completes the characterization. Our main results are as follows:

Theorem A. *The intermediate space* $(L^1, L \log^+ L)_{\theta, q; K}$, $0 < \theta < 1$, $1 \leq q \leq \infty$, *consists of all integrable functions* f *on* $[0, 1]$ *for which the norm*

$$\|f\| = \left\{ \int_0^1 [t(\log 1/t)^{\theta - 1/q} f^{**}(t)]^q dt/t \right\}^{1/q}$$

is finite.

Corollary B. *When* $0 < \theta < 1$, *we have* $(L^1, L \log^+ L)_{\theta, 1; K} = L (\log^+ L)^\theta$, *with equivalent norms.*

Corollary C. *When* $1 \leq q \leq \infty$, *we have* $(L^1, L \log^+ L)_{1/q, q; K} = L^{1q}$, *with equivalent norms.*