The distribution of square-full integers

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1. Introduction

It is well-known that a positive integer n is called *square-full*, if in the canonical representation of n into prime powers each exponent is ≥ 2; or equivalently, if each prime factor of n occurs with multiplicity at least two. The integer 1 is also considered to be square-full. Let L denote the set of square-full integers and $l(n)$ denote the characteristic function of the set L, that is, $l(n) = 1$ or 0 according as $n \in L$ or $n \notin L$. Let $L(x)$ denote the enumerative function of the set L, that is, $L(x) = \sum_{n \leq x} l(n)$, where x is a real variable ≥ 1.

In 1934, P. Erdös and G. Szekeres (cf. [7], § 2) proved the following asymptotic formula, using elementary methods:

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}). \tag{1.1}$$

A simple proof of this result has been given later by A. Sklar [12]. In 1954, P. T. Bateman [1] improved the result (1.1) by means of the Euler Maclaurin sum formula to

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}), \tag{1.2}$$

where

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{(t - [t])}{t^{s+1}} \, dt, \quad (s > 0, \ s \neq 1); \tag{1.3}$$

and he remarked that, by more delicate methods, it is possible to sharpen the error term in (1.2) to $O(x^{1/6} \log^2 x)$.