Invariant Subspaces and Weighted Polynomial Approximation

James E. Brennan

1. Introduction

Let μ be a finite positive Borel measure defined and having compact support in the complex plane $\mathbf{C}$. Assume that μ is not a point mass. Let z denote the complex identity function and let $\mathcal{P}$ stand for the polynomials in z. For each $p, 1 \leq p < \infty$, set $H^p(d\mu)$ equal to the closure of $\mathcal{P}$ in $L^p(d\mu)$. In this paper we ask: Does $H^p(d\mu)$ have at least one closed subspace, other than itself and $\{0\}$, which is invariant under multiplication by z? The answer is known to be yes when $p > 2$ and yes in certain cases when $p \leq 2$. When $p = 2$ the question is especially intriguing, since then it is equivalent to the invariant subspace problem for subnormal operators on Hilbert space. Our main objective here is to answer it for certain measures ψdA which are absolutely continuous with respect to planar Lebesgue measure A.

We begin with a few simple observations. If $H^p(\psi dA) = L^p(\psi dA)$ and W is any measureable set with $0 < \int_W \psi dA < \int \psi dA$ then $S = \{f \in H^p(\psi dA): f = 0$ a.e. $- \psi dA$ on $W\}$ is a nontrivial closed subspace invariant under multiplication by z. If $H^p(\psi dA) \neq L^p(\psi dA)$ (and only then) it may happen that there is a point $\zeta \in \mathbf{C}$ such that the map $f \to f(\zeta)$ can be extended from $\mathcal{P}$ to a bounded linear functional on $H^p(\psi dA)$. A linear functional on $H^p(\psi dA)$ associated to a point ζ in this way is called a bounded evaluation for $H^p(\psi dA)$. By taking $S(\zeta)$ to be the closure in $H^p(\psi dA)$ of the polynomials vanishing at ζ, we obtain a closed subspace which is invariant under multiplication by z and, since $(z - \zeta) \in S(\zeta)$ and $1 \notin S(\zeta)$, it is nontrivial. In some cases it can be shown that either $H^p(\psi dA)$ has a bounded evaluation or else $H^p(\psi dA) = L^p(\psi dA)$, thereby assuring the existence of a z-invariant subspace in $H^p(\psi dA)$.

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* This work was partially supported by NSF Grant No. G.P. 19533.