On the regularity of the distribution of the Fekete points of a compact surface in R^n

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1. Introduction

The question to be considered in this note can be stated in terms of classical physics. Let a finite number of equal point charges find a stable equilibrium distribution within an isolated conductor. Then how well does this distribution approximate the equilibrium distribution of the same total charge?

We let S be a compact $(n-1)$-dimensional surface of class $C^{1,\alpha}$ in $\mathbf{R}^n$, which is supposed to be the common boundary of two domains. Only the case $n \geq 3$ will be considered, although the method of proof works also in the plane for a simple closed curve of the same regularity. For $N > 1$ we let $(z_{N1}, \ldots, z_{NN})$ be a system of Fekete points of S, i.e., an N-tuple of points in S which among all N-tuples of points in S minimizes the energy

$$\frac{1}{N^2} \sum_{i \neq j}\sum \frac{1}{|z_{Ni} - z_{Nj}|^{n-2}}$$

of the mass distribution μ_N consisting of point masses $1/N$ at each z_{Ni}. Here we omit the infinite energy of each point mass. The Fekete points need not be uniquely determined by S and N.

Let λ be the equilibrium distribution of total mass 1 in S, to which μ_N will be compared. In two dimensions several estimates of $\mu_N - \lambda$ have been made. In that case our method yields the result that any subarc B of the curve satisfies

$$|\mu_N(B) - \lambda(B)| \leq \text{const.}\ \frac{\sqrt{\log N}}{\sqrt{N}}\ .$$

This should be compared to Kleiner's bound, const. $\log N/\sqrt{N}$ in [1]. For analytic curves Pommerenke has given much sharper estimates in for example [3], where further references can be found.