Projective Planes with a Condition on Projectivities

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Introduction

Let π be a projective plane, L, M two lines of π, $\sum$ the set of points on L excluding the point of intersection of L and M, and I a point of π lying on neither L nor M. If J is any point of π not lying on L or M we can define a permutation r of $\sum$ by first projecting L onto M through I and then projecting M back onto L through J. We denote the set of permutations r obtained from every such J by R. R has the following properties which are characteristic of projective planes.

I $1 \in R$

II If $\alpha, \beta, \gamma, \delta \in \sum$, $\alpha \neq \beta$, $\gamma \neq \delta$ there is a unique member r of R with the properties $r(\alpha) = \gamma$, $r(\beta) = \delta$

III The relation $\sim$ on R defined by $r \sim s$ if $r = s$ or $r(\alpha) \neq s(\alpha)$ for every $\alpha \in \sum$, is an equivalence relation. Each equivalence class is sharply transitive on $\sum$, i.e. if $\alpha, \beta \in \sum$ each class contains exactly one member r with $r(\alpha) = \beta$.

A set R of permutations on a set $\sum$ which satisfies I, II and III will be called sharply doubly transitive and if G is a group of permutations on $\sum$ which contains R we call R a sharply doubly transitive subset of G.

A consequence of the axiom of Pappus in a projective plane is that any projectivity of a line which transposes two points of the line must act as an involution on the line. The converse of this result follows from a theorem of J. Tits [4]. The unrestricted group of projectivities on a line in a projective plane acts triply transitively on the points of that line and it is a simple consequence of the result of Tits that a triply transitive group in which only involutions transpose symbols must be a group $PGL(2, F)$ of all bilinear transformations

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