A Banach space with basis constant > 1.

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If a Banach space has a Schauder basis β, then $\beta_K = \sup_{x,n} \|\sum_{i=1}^{n} a_i e^i\|/\|x\|$ exists, where $x = \sum_{i=1}^{\infty} a_i e^i$. Inf β_K taken over all β is called the basis constant of the Banach space. It is obvious that if the Banach space B has the basis constant p, then every finite-dimensional subspace C of B can be approximated by subspaces D_n of B – by approximating a set of basis vectors of C with vectors of finite expansions in some basis – such that each D_n can be embedded into a finite-dimensional subspace E_n of B, onto which there is a projection from B of norm arbitrarily close to p.

In this paper we construct a separable infinite-dimensional Banach space B with a two-dimensional subspace C_1 with the following properties: There is a $p > 1$ such that, if D is a two-dimensional subspace of B sufficiently close to C_1 and E is a finite-dimensional subspace of B containing D, then there is no projection from B onto E of norm $\leq p$. Thus the basis constant of this Banach space is $\geq p$. This seems to be by now the strongest result in negative direction on the well-known basis problem. The previously strongest result seems to be Gurarii's example of a Banach space where $\beta_K > 1$ for every β. (See Singer [1] pp. 218–42.)

We now start by giving a general and somewhat unprecise description of the ideas behind the construction and of the problems we meet. We consider a two-dimensional subspace C_1 of $l_\infty(\Gamma)$, where Γ is the set of pairs of positive integers. We assume that the projection constant of C_1 is > 1. Now our first ambition will be to embed C_1 in a larger space E_1, such that there is no projection of norm close to 1 from E_1 onto spaces close to C_1 and such that no subspace C_2 of E_1 containing a subspace of E_1 sufficiently close to C_1 has a projection constant near to 1. However, if we try to do this we have to get control of quite many linear spaces. In order to describe how we obtain the necessary simplifications, we give now a description of the way we estimate norms of projections.