A remark on embedding theorems for Banach spaces of distributions

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1. Introduction and results

This note is more or less an appendix to the paper [9]. We use the notions of [9] and recall some of them. R_n is the n-dimensional Euclidean space: $x = (x_1, \ldots, x_n) \in R_n$. $S(R_n)$ is the Schwartz space of rapidly decreasing (complex) infinitely differentiable functions, $S'(R_n)$ is the dual space of tempered distributions (with the strong topology). F is the Fouriertransformation in $S'(R_n)$, F^{-1} the inverse Fouriertransformation. We use special systems of functions $\{\varphi_k\}_{k=0}^{\infty}$ (see [9], 4.2.1) with

1. $\varphi_k(x) \in S(R_n)$, $F\varphi_k(x) \geqq 0$; $k = 0, 1, 2, \ldots$
2. $\exists\, N$; $N = 1, 2, \ldots$; with $\operatorname{supp} F\varphi_k \subset \{\xi | 2^{k-N} \leqq |\xi| \leqq 2^{k+N}\}$ for $k = 1, 2, \ldots$; $\operatorname{supp} F\varphi_0 \subset \{\xi | |\xi| \leqq 2^N\}$; (supp denotes the support of a function).
3. $\exists\, c_1 > 0$ with $c_1 \leqq (\sum_{j=0}^{\infty} F\varphi_j)(\xi)$;
4. $\exists\, c_2 > 0$ with $|(D^\alpha F\varphi_k)(\xi)| \leqq c_2 |\xi|^{-|\alpha|}$ for $0 \leqq |\alpha| \leqq [n/2] + 1$; $k = 1, 2, \ldots$

The most important system of functions of this type is the following. We consider a function $\varphi(x) \in S(R_n)$; $(F\varphi)(x) \geqq 0$;

$$\operatorname{supp} F\varphi \subset \{\xi | 2^{-N} \leqq |\xi| \leqq 2^N\}; \quad (F\varphi)(\xi) > 0 \quad \text{for} \quad 1/\sqrt{2} \leqq |\xi| \leqq \sqrt{2}\,. \tag{1}$$

It is not difficult to see that the functions $\varphi_k(x)$ with

$$(F\varphi_k)(\xi) = (F\varphi)(2^{-k}\xi); \quad k = 1, 2, \ldots; \tag{2}$$

by suitable choice of $\varphi_0(x)$ are a system of the described type.

Now we define the spaces $F^s_{p,q} = F^s_{p,q}(R_n)$ and $B^s_{p,q} = B^s_{p,q}(R_n)$. Let $-\infty < s < \infty$; $1 < p < \infty$; $1 < q < \infty$; $\{\varphi_k\}_{k=0}^{\infty}$ is a system of the described type. We set

$$F^s_{p,q} = \left\{ f | f \in S'(R_n),\ \|\{f * \varphi_k\}\|_{L_p(l^s_q)} = \left[\int\limits_{R_n} \Big(\sum_{k=0}^{\infty} 2^{sqk} |(f * \varphi_k)(x)|^q\Big)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} < \infty \right\}. \tag{3}$$