Linear measure on plane continua of finite linear measure

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1. Introduction

Let Λ be one-dimensional Hausdorff measure in $\mathbf{C}$. Let B be a continuum in $\mathbf{C}$ of finite linear measure, i.e., $\Lambda(B)<\infty$. Denote the components of the complement of B in the Riemann sphere by $\{V_j\}$; each V_j is simply connected. Let $f_j\colon \mathbf{D}\to V_j$ be a Riemann map, where $\mathbf{D}$ is the open unit disk.

Theorem.

$$2\Lambda(B)=\sum_j \int_0^{2\pi} |f_j'(e^{i\theta})|\, d\theta.$$

The problem of establishing this identity was raised by Ch. Pommerenke in a letter to the author. The proof has two parts. First we show that the j^{th} integral in the theorem is equal to the integral with respect to Λ of the multiplicity function of f_j; this is (I). The second part, (II), is to show that the sum of the multiplicity functions is equal to 2 a.e. on B with respect to Λ.

After the proof of the theorem we shall indicate a generalization. This is a decomposition of the restriction of Λ to B as a sum of measures on the boundaries of the V_j.

Finally I want to thank Professor Ch. Pommerenke for writing to me about this problem. A related inequality had been treated in [1].

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