Interval estimates

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§ 1. Introduction

In this paper we prove that there exists an absolute constant $l>0$ such that, for every univalent H^1 function f in the open unit disk D and every $z_0 \in D$, there are $\vartheta \in \mathbf{R}$ and ε, $l(1-|z_0|) \leqq \varepsilon \leqq \pi$, such that

$$f(z_0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt.$$

Let f be a holomorphic function in the open unit disk D which belongs to the Hardy class H^1 ([4]). According to [1] and [2], every value $f(z_0)$, $|z_0|<1$, is of the form

$$f(z_0) = \frac{1}{|I|} \int_I f(e^{i\vartheta})\, d\vartheta,$$

where I is an interval on the unit circle with length $|I|$, $0<|I| \leqq 2\pi$. A sketch of the proof is given in Prop. 1, § 2 below. The proof does not provide information on the size or the location of the interval I. Extensions of the previous result in [5, 6] are related to BMO, measures and holomorphic mappings in several variables; still they do not contain quantitative information on the size of I. Some preliminary quantitative results concerning univalent functions can be found in [7] and [8]. Their proof makes use of the classical distortion theorems and especially of the 1/4-Koebe theorem.

The purpose of the present paper is to furnish a brief and complete presentation of the above quantitative results on univalent functions; the general H^1 case is, as far as I know, still open.

The main result, thus, states that if f is H^1 and univalent then $|I| \geqq 2l(1-|z_0|)$, where $l>0$ is an absolute constant independent of f and z_0. In the particular case where $f(z) = \log(1-z)$, the length $|I|$ is exactly of the order of $(1-|z_0|)$; however, I do not know the best value of the constant l.