A new proof of a sharp inequality concerning the Dirichlet integral

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If f is an analytic function defined in the unit disk, Δ, let

$$\mathscr{D}(f) = \left(\int_{\Delta} |f'(z)|^2 \frac{dx\,dy}{\pi}\right)^{1/2} \quad (z = x+iy)$$

be the Dirichlet integral of f. In this note, we will give a new proof of the following theorem.

Theorem 1 [4]. *There is a constant $C<\infty$ such that if f is analytic on Δ, $f(0)=0$, and $\mathscr{D}(f)\leqq 1$ then*

$$\int_0^{2\pi} e^{|f(e^{i\theta})|^2}\, d\theta \leqq C.$$

See [4] for motivation. Following a suggestion of L. Carleson, we give a proof of this theorem based on an unpublished result of A. Beurling. In Section 1, we give a proof of a version of Beurling's theorem. In Section 2, we use Beurling's theorem to reduce our problem to an integral inequality due to J. Moser [7]. Finally, in Section 3, we give a proof of Moser's inequality. We would like to thank P. Jones for helpful discussions.

1. Beurling's theorem

If f is analytic on Δ, let γ_t be the level curves of f defined by $\gamma_t=\{z\in\Delta: |f(z)|=t\}$ and let $|f(\gamma_t)|$ denote the length of the image of these curves under the map f. In other words, $|f(\gamma_t)|=\int_{\gamma_t} |f'(z)|\,|dz|$ where $|dz|$ denotes arc length. If E is a subset of the complex plane, let cap(E) denote the logarithmic capacity of E.

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