The two-sided complex moment problem

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Hamburger's theorem asserts that a function $\varphi\colon \mathbf{N}_0\to\mathbf{R}$ is the moment sequence of a measure on the real line if and only if φ is positive definite in the sense that the kernel $(n, m)\to\varphi(n+m)$ is positive semidefinite. (See [1], Theorem 6.2.2). The corresponding two-sided problem, consisting in characterizing those functions $\varphi\colon \mathbf{Z}\to\mathbf{R}$ which are the two-sided moment sequences of measures μ on $\mathbf{R}\setminus\{0\}$ in the sense of

$$\varphi(n) = \int x^n \, d\mu(x), \quad n\in\mathbf{Z},$$

has a similar solution: Such functions φ are precisely those which are positive definite in the sense that the kernel $(n, m)\to\varphi(n+m)$ is positive semidefinite. This was shown in [4]; see [1], Theorem 6.4.1, for a simple proof.

The complex moment problem, a natural analogue of the moment problem solved by Hamburger, requires the characterization of those functions $\varphi\colon \mathbf{N}_0^2\to\mathbf{C}$ which are complex moment sequences of measures μ on $\mathbf{C}$ in the sense that

$$\varphi(n, m) = \int z^n \bar{z}^m \, d\mu(z), \quad (n, m)\in\mathbf{N}_0^2.$$

In this case the pertinent concept of positive definiteness arises by considering the semigroup $(\mathbf{N}_0^2, +)$ with the involution (*) given by $(n, m)^* = (m, n)$ and agreeing to call a function $\varphi\colon \mathbf{N}_0^2\to\mathbf{C}$ positive definite if the kernel $(s, t)\to\varphi(s+t^*)$ on $\mathbf{N}_0^2\times\mathbf{N}_0^2$ is positive semidefinite. While every complex moment sequence is positive definite, there exist positive definite functions on $\mathbf{N}_0^2$ which are not complex moment sequences ([1], Theorem 6.3.5).

Observe that [1], Theorem 6.1.10 implies the following somewhat roundabout solution of the complex moment problem: A function $\varphi\colon \mathbf{N}_0^2\to\mathbf{C}$ is a complex moment sequence if and only if $\sum_{n,m} c_{n,m}\varphi(n, m)\geqq 0$ for each $(c_{n,m})\in\mathbf{C}^{(\mathbf{N}_0^2)}$ such that $\sum_{n,m} c_{n,m} z^n \bar{z}^m \geqq 0$ for all $z\in\mathbf{C}$.

Like Hamburger's moment problem, the complex moment problem has a two-sided companion, the problem of characterizing two-sided complex moment se-