On spaces of Triebel—Lizorkin type

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0. Introduction

In this note we study certain spaces of distributions $F_p^{sq}=F_p^{sq}(\mathbf{R}^n)$ where s real, $0<p, q\leqq\infty$. They are intimately related to certain spaces studied by Triebel [10] and Lizorkin [5] (cf. also [6]) when $1<p, q\leqq\infty$. Our main result is a certain equivalence theorem (see Sec. 3) which says that the spaces do not depend on the special sequence of testfunctions $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$ entering in their definition. This extends Triebel's corresponding result. But we have to give an entirely new proof, relying on two deep results by Fefferman & Stein: 1° their real variable characterization of the Hardy classes H_p [1], 2° their sequence valued version of the Hardy & Littlewood maximal theorem [2]. (Incidentally it follows from [1] that $F_p^{02}=H_p$ if $0<p<\infty$ while as $F_\infty^{02}=$ B. M. O.!) As an application we prove (see Sec. 5) a multiplier theorem of the Mikhlin type, extending the one by Triebel and Lizorkin. We also give (see Sec. 6) an application to approximation theory related to a theorem of Freud's [3]. Finally we briefly indicate (see Sec. 7) how the result might be extended to the case of a Riemannian manifold.

1. Definitions

By L_p where $0<p\leqq\infty$ we denote the space of measurable functions $f=f(x)$ $(x\in\mathbf{R}^n)$ such that

$$\|f\|_{L_p}=\left(\int|f(x)|^p\,dx\right)^{1/p}<\infty.$$

By l^q where $0<q\leqq\infty$ we denote the space of sequences $\mathbf{t}=\{t_\nu\}_{\nu\in\mathbf{Z}}$ such that

$$\|\mathbf{t}\|_{l^q}=\left(\sum_{\nu\in\mathbf{Z}}|t_\nu|^q\right)^{1/q}<\infty.$$

We consider also spaces of sequence valued measurable functions $L_p(l^q)$ and $l^q(L_p)$, defined in the obvious way. If $1\leqq p, q\leqq\infty$ these are all Banach spaces, in the general case only quasi-Banach space.